Noncompactness for the σ_2-Yamabe problem

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The Yamabe equation and the bubble family on the round sphere

Classic Yamabe problem

On a closed (M^n, g), $n \geq 3$, set $\tilde{g} = u^{4/(n-2)} g$ and seek $u > 0$ with $R_{\tilde{g}}$ constant.

$$L_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \qquad L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$$

$$L_{\tilde{g}}(\varphi) = u^{-\frac{n+2}{n-2}} L_g(u\varphi) \qquad \text{(conformal covariance)}$$

Round sphere and its conformal orbit

Normalize the round metric by $R_{g_{S^n}} = 4n(n-1)$. For every conformal automorphism Φ of S^n, $\Phi^* g_{S^n} = u_\Phi^{4/(n-2)} g_{S^n}$ is again round; this already gives a noncompact family of Yamabe solutions. In stereographic coordinates, the same family is

$$-\Delta U = n(n-2) U^{\frac{n+2}{n-2}}, \qquad U_{\lambda,\xi}(x) = \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}}, \qquad \lambda > 0, \quad \xi \in \mathbb{R}^n.$$

The Yamabe problem is variational

$$Q_g(u) = \frac{\int_M (a_n |\nabla u|_g^2 + R_g u^2)\, dv_g}{\left(\int_M |u|^{2^*}\, dv_g\right)^{2/2^*}}, \qquad Y(M,[g]) = \inf_{u\neq 0} Q_g(u)$$

$$2^* = \frac{2n}{n-2}, \qquad a_n = \frac{4(n-1)}{n-2}.$$

Sphere threshold

If the infimum is achieved, its Euler–Lagrange equation is the Yamabe equation. The only obstruction is concentration:

$$\text{one bubble} \quad \rightsquigarrow \quad \text{energy } Y(S^n).$$

$$Y(M,[g]) < Y(S^n) \Longrightarrow \text{no concentration} \Longrightarrow \text{minimizer} \Longrightarrow \text{Yamabe metric.}$$

How strict inequality is proved

- **Yamabe (1960):** variational formulation.
- **Trudinger (1968):** corrected the limiting argument in a restricted regime.
- **Aubin (1976):** local bubble test functions detect nonzero Weyl curvature.
- **Schoen (1984):** Green function expansion plus positive mass handles the remaining cases.

Compactness breaks precisely after dimension 24

Compactness

- **Question.** Are normalized positive solutions uniformly bounded?
- **Compactness.** Uniform L^∞-bounds give precompactness in $C^{2,\alpha}$.
- **Progress.** Schoen (1989); Druet (2004); Marques (2005); Li–Zhang (2004–2005).
- **Sharp range.** Khuri–Marques–Schoen (2009): $3 \le n \le 24$, away from the round-sphere conformal class.

Noncompactness

- **Round sphere.** Conformal symmetry already produces a noncompact family.
- **Brendle (2008).** Fixed non-round metrics with blow-up for $n \ge 52$.
- **Brendle–Marques (2009).** Counterexamples for $25 \le n \le 51$.
- Hence noncompact examples exist in every dimension $n \ge 25$.

$$\|u_j\|_{L^\infty} \to \infty, \qquad \mu_j = u_j(x_j)^{-\frac{2}{n-2}}, \qquad \mu_j^{\frac{n-2}{2}} u_j\Big(\exp_{x_j}(\mu_j y)\Big) \longrightarrow U_{1,0}(y).$$

Brendle's perturbation of sphere metric $U^{4/(n-2)}dx^2$

Fixed background metric: small perturbations of Euclidean metric

One smooth metric is fixed by summing local Weyl perturbations whose centers converge to one point.

$$g = e^h, \qquad h_{ik}(x) = \sum_{N=N_0}^{\infty} h_{ik}^{(N)}(x)$$

$$h_{ik}^{(N)}(x) = \eta\left(\rho_N^{-1}|x-y_N|\right)\mu_N\lambda_N^{2d} f\left(\lambda_N^{-2}|x-y_N|^2\right) H_{ik}(x-y_N), \qquad H_{ik}(z) = \sum_{p,q=1}^{n} W_{ipkq}z_p z_q.$$

$$y_N = N^{-1}e_1, \qquad \lambda_N = 2^{-N/2}, \qquad \mu_N = 2^{-N}, \qquad \rho_N = (2N)^{-2}.$$

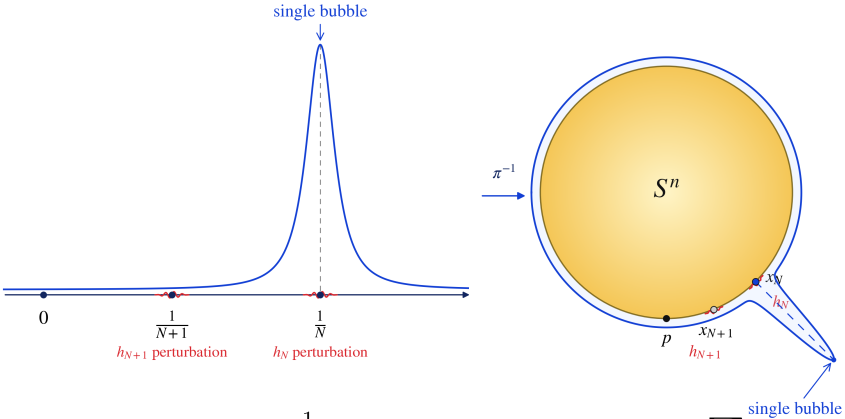
Perturbation of AT bubble as conformal factor

To find $v_{\lambda,\xi}^{4/(n-2)}g$ with constant scalar curvature where $v_{\lambda,\xi} = U_{\lambda,\xi} + \phi_{\lambda,\xi}$, the solution of

$$L_g v = n(n-2)v^{\frac{n+2}{n-2}}.$$

Brendle's ansatz on $\mathbb{R}^n$ and S^n

One bubble over a fixed locally perturbed metric



$$\pi(p) = 0, \qquad \pi(x_N) = \frac{1}{N} e_1, \qquad x_N \longrightarrow p, \qquad g = e^h, \quad h = \sum_{N \gg 1} h_N.$$

Finite-dimensional reduction: the PDE becomes $n+1$ equations

Bubble kernel and orthogonality

With $\mathcal{E}_g(v) = L_g v - n(n-2)v^{\frac{n+2}{n-2}}$, set $Z_0 = \partial_\lambda U_{\lambda,\xi}$ and $Z_j = \partial_{\xi_j} U_{\lambda,\xi}$.

$$v_{\lambda,\xi} = U_{\lambda,\xi} + \phi_{\lambda,\xi}, \qquad \phi_{\lambda,\xi} \perp \operatorname{span}\{Z_0, Z_1, \ldots, Z_n\}.$$

Projected equation

$$\begin{aligned} \Pi^{\perp}\mathcal{E}_g(U_{\lambda,\xi} + \phi_{\lambda,\xi}) = 0 \\ \Longrightarrow \quad \phi_{\lambda,\xi} = \phi(\lambda,\xi). \end{aligned}$$

Kernel obstruction

$$\Pi\mathcal{E}_g(U_{\lambda,\xi} + \phi_{\lambda,\xi}) = \sum_{j=0}^{n} c_j(\lambda,\xi) Z_j.$$

Variational identity and reduced model

$$\begin{aligned} c_j(\lambda,\xi) = 0 \ (0 \le j \le n) \quad &\Longleftrightarrow \quad \nabla_{(\lambda,\xi)}\mathcal{F}_g(\lambda,\xi) = 0, \\ \mathcal{F}_g(\lambda,\xi) = C + \varepsilon_N F(\lambda,\xi) + o(\varepsilon_N), \quad &\quad F \text{ has a strict local minimum.} \end{aligned}$$

The σ_k-Yamabe equation

Conformal Schouten tensor

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)} g\right), \qquad \tilde{g} = e^{-2w} g$$
$$A_{\tilde{g}} = A_g + \nabla_g^2 w + dw \otimes dw - \frac{1}{2}|\nabla_g w|^2 g.$$

Constant σ_k-curvature equation

$$\sigma_k\left(\lambda(\tilde{g}^{-1} A_{\tilde{g}})\right) = C, \qquad \lambda(\tilde{g}^{-1} A_{\tilde{g}}) \in \Gamma_k^+ = \{\lambda \in \mathbb{R}^n : \ \sigma_1 > 0, \cdots, \sigma_k > 0\}$$
$$\Longleftrightarrow \quad \sigma_k\left(\lambda\left(g^{-1}\left[A_g + \nabla_g^2 w + dw \otimes dw - \tfrac{1}{2}|\nabla_g w|^2 g\right]\right)\right) = Ce^{-2kw}.$$

For $k=1$, this is the Yamabe equation

$$\sigma_1(g^{-1} A_g) = \frac{R_g}{2(n-1)}, \qquad k = 1 \quad \Longrightarrow \quad R_{\tilde{g}} = \text{constant}.$$

Why $k = 2$ is the first nonlinear Yamabe problem

Nonlinear ellipticity is part of the equation

$$\sigma_2(B) = \frac{1}{2}\langle T_1(B), B\rangle, \qquad T_1(B) = \sigma_1(B)I - B$$

$$\text{ellipticity} \quad \Longleftrightarrow \quad T_1(B) > 0 \quad \Longleftarrow \quad \lambda(B) \in \Gamma_2^+.$$

For $k = 2$, the equation is still variational, LCF & non-LCF

$$\mathcal{E}_2(g) = \int_M \sigma_2(g^{-1}A_g)\, dV_g, \qquad n \neq 4$$

$$\delta\mathcal{E}_2|_{[g]} = 0 \quad \Longleftrightarrow \quad \sigma_2(g^{-1}A_g) = \text{constant.}$$

Main difficulty in blow-up constructions

$$\text{control } T_1 > 0 \quad + \quad \text{solve the equation} \quad + \quad \text{stay inside } \Gamma_2^+.$$

Existence theory for σ_k-Yamabe

The equation

$$\sigma_k\Big(\lambda(g_w^{-1}A_{g_w})\Big) = \text{constant}, \qquad g_w = e^{-2w}g, \qquad \lambda(g_w^{-1}A_{g_w}) \in \Gamma_k^+.$$

LCF theory

Guan–Wang (2003)
Conformal flow and existence in the locally conformally flat setting.

A. Li–Y.Y. Li (2003, 2005)
Liouville theorems, Harnack inequalities, local estimates, and LCF existence.

Non-LCF / high-dimensional theory

Chang–Gursky–Yang (2002)
Dimension four σ_2-geometry; admissibility and total σ_2-curvature.

Sheng–Trudinger–Wang (2007); Ge–Wang (2006)
Existence results for higher-order curvature Yamabe equations beyond the LCF case.

Compactness theory for σ_k-Yamabe

Compactness means uniform control of conformal factors

$$g_j = u_j^{\frac{4}{n-2k}} g, \qquad \sigma_k(g_j^{-1}A_{g_j}) = \text{constant}, \qquad \lambda(g_j^{-1}A_{g_j}) \in \Gamma_k^+$$
$$\sup_M u_j \le C \quad \Longrightarrow \quad \{u_j\} \text{ precompact modulo conformal symmetries.}$$

LCF compactness

A. Li–Y.Y. Li (2003, 2005)
Harnack and compactness theory in the LCF case, with the round sphere as the conformal-orbit exception.

High-k compactness

Gursky–Viaclovsky (2007); Trudinger–Wang (2005, 2009)
For $n/2 < k < n$, sharp Harnack inequalities and compactness of normalized admissible conformal metrics.

Li–Nguyen (2014)
Fully nonlinear compactness under Ricci lower bounds; covers the borderline $k = n/2$ range.

$$k = 2 < n/2 \quad (n \ge 5): \qquad \text{the non-LCF compactness picture is much less rigid.}$$

Our theorem: antipodal noncompactness for σ_2

Theorem (Deng–Lu–Wei, $\mathrm{arXiv:2607.11067}$)

Let $n \geq 27$. There exist a smooth antipodally symmetric, non-locally conformally flat metric g_0 on S^n and positive functions $v_N \in C^\infty(S^n)$ such that $g_{v_N} = v_N^{8/(n-4)} g_0$ are admissible constant-σ_2 metrics.

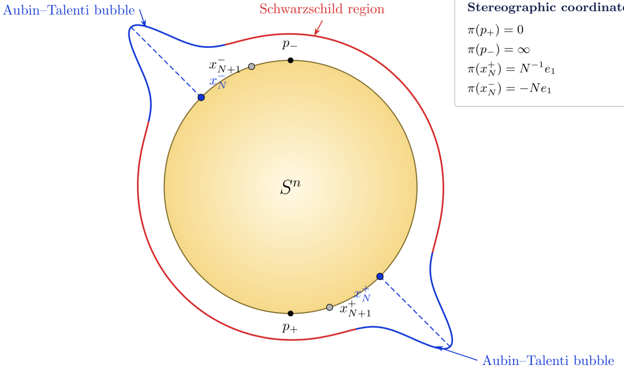
Constant σ_2-curvature and blow-up

$$\sigma_2\left(g_{v_N}^{-1} A_{g_{v_N}}\right) = \frac{1}{4}\binom{n}{2}, \qquad \mu\left(g_{v_N}^{-1} A_{g_{v_N}}\right) \in \Gamma_2^+,$$

$$v_N \circ J = v_N, \qquad J(x) = -x, \qquad \sup_{S^n} v_N \to \infty, \qquad \mathrm{Blow}(v_N) = \{p_+, p_-\}.$$

Our ansatz pairs two antipodal bubbles through a Schwarzschild bridge

One antipodally symmetric solution on S^n; no topology change



Stereographic coordinates

$\pi(p_+) = 0$

$\pi(p_-) = \infty$

$\pi(x^+_N) = N^{-1}e_1$

$\pi(x^-_N) = -Ne_1$

two Aubin–Talenti cores at $x^{\pm}_N$ $\overset{\text{smooth matching}}{\longleftrightarrow}$ Schwarzschild bridge, $\quad \bar{u}_N \circ J = \bar{u}_N, \quad J(x) = -x.$

Why Brendle's one-bubble ansatz can lose ellipticity for σ_2

Newton tensor for Bubble $U_{\lambda,0}(x) = \left(\frac{\lambda}{\lambda^2+r^2}\right)^{(n-2)/2}$,

$$T_1\Big(B[U_{\lambda,0}]\Big) = \frac{2(n-1)q_B}{r^2} I, \qquad \frac{q_B}{r^2} = \frac{\lambda^2}{(\lambda^2+r^2)^2}.$$

Far from the core $r \gg \lambda$

$$q_B = \frac{\lambda^2}{r^2} + O\left(\frac{\lambda^4}{r^4}\right), \qquad \frac{q_B}{r^2} \sim \lambda^2 r^{-4}.$$

Add effects of background metric

$$T_1\Big(B[U_{\lambda_N,0};h]\Big) = \frac{2(n-1)q_B}{r_N^2} I + E_h, \qquad |E_h| \lesssim |D^2 h| + |Dh|^2 + r_N^{-1}|Dh| + r_N^{-2}|h| \lesssim \alpha.$$

Ellipticity requires a relative estimate:

$$|E_h| \ll \frac{q_B}{r_N^2} \sim \lambda_N^2 r_N^{-4} \approx 2^{-N} \to 0 \qquad \text{as } N \to \infty,$$

which is impossible!

The σ_2-Schwarzschild mode: an exact $\sigma_2 = 0$ model

Conformal Schwarzschild factor

$$a = \frac{n-4}{4}, \qquad S_\lambda(r) = (2\lambda)^a\Big(1 + r^{-2a}\Big), \qquad g_{\Sigma,\lambda} = S_\lambda^{8/(n-4)}\delta.$$

Radial variables

$$p_\Sigma(r) = \frac{r^{-2a}}{1 + r^{-2a}}, \qquad q_\Sigma(r) = p_\Sigma(1 - p_\Sigma), \qquad \dot{p}_\Sigma = -2aq_\Sigma, \quad \dot{f} := r\partial_r f.$$

Direct computation of the Schouten endomorphism

$$B_\Sigma := g_{\Sigma,\lambda}^{-1} A_{g_{\Sigma,\lambda}} = \frac{q_\Sigma}{r^2}\Big(-(n-2)e_r \otimes e_r + 2\Big(I - e_r \otimes e_r\Big)\Big).$$

$$\lambda_R(B_\Sigma) = -\frac{(n-2)q_\Sigma}{r^2}, \qquad \lambda_T(B_\Sigma) = \frac{2q_\Sigma}{r^2} \quad \text{with multiplicity } n-1.$$

The mode has zero σ_2-curvature

$$\sigma_2(B_\Sigma) = (n-1)\lambda_R\lambda_T + \frac{(n-1)(n-2)}{2}\lambda_T^2 = -2(n-1)(n-2)\frac{q_\Sigma^2}{r^4} + 2(n-1)(n-2)\frac{q_\Sigma^2}{r^4} = 0.$$

Positive Newton tensor, note q_Σ doesn't contain bad λ term

$$T_1(B_\Sigma) = \sigma_1(B_\Sigma)I - B_\Sigma, \qquad \tau_R(T_1) = \frac{2(n-1)q_\Sigma}{r^2}, \qquad \tau_T(T_1) = \frac{(n-2)q_\Sigma}{r^2} > 0.$$

Sharp h-estimates keep the Newton tensor positive

Bubble core ellipticity: compare with the bubble margin

$$T_1(B_B) = 2(n-1)\frac{q_B}{r^2}I, \qquad \frac{q_B}{r^2} = \frac{\lambda_N^2}{(\lambda_N^2+r^2)^2},$$

$$\mathcal{E}_h^B := |D^2h| + r^{-1}|Dh| + r^{-2}|h| + |Dh|^2 \le \eta_N \frac{q_B}{r^2}, \qquad \eta_N = o(1),$$

$$\text{since } |D^j h_N^{\pm}| \le C_j \mu_N (\epsilon_N + r)^{10-j} \quad (0 \le j \le 2), \qquad \frac{r^2}{q_B}\mathcal{E}_h^B \le C\epsilon_N^{\kappa_f}.$$

Schwarzschild neck ellipticity: compare with the neck margin

$$p_\Sigma = \frac{r^{-2a}}{1+r^{-2a}}, \qquad q_\Sigma = p_\Sigma(1-p_\Sigma), \qquad T_1(B_\Sigma) \ge c\frac{q_\Sigma}{r^2}I,$$

$$\mathcal{E}_h^\Sigma := |D^2h| + r^{-1}|Dh| + r^{-2}|h| + |Dh|^2 \le \eta_N \frac{q_\Sigma}{r^2}, \qquad \eta_N \le C\left(\delta_*(N_0) + \epsilon_N^{\kappa_f}\right) = o(1),$$

$$\text{since } \sum_{j=0}^{2} r^{j-2}|D^j h_{\mathrm{rem,N}}| \le \delta_*(N_0) r^{2a-2},$$

$$\mathbf{T}_h[\bar u] \ge \mathbf{T}_0[\bar u] - C\eta_N \frac{q}{r^2}I \ge c\frac{q}{r^2}I > 0, \qquad q = q_B \text{ or } q_\Sigma.$$

Transition: reduce the bridge to one scalar ODE

Work after Kelvin inversion: Schwarzschild → bubble

$$p[u] := -\frac{2}{n-4}\partial_{\log r}\log u, \qquad q = p(1-p), \qquad \mathcal{V} = r^a u, \qquad a = \frac{n-4}{4}, \qquad r = R_\lambda^+ e^s, \qquad |s| \le L, \qquad \delta_\lambda = \lambda^{2-8/n}.$$

Endpoint slopes are both of size δ_λ

$$p_\Sigma(s) = \frac{\delta_\lambda e^{-2as}}{1+\delta_\lambda e^{-2as}} = \delta_\lambda e^{-2as} + O_L(\delta_\lambda^2), \qquad p_B(s) = \frac{\delta_\lambda e^{2s}}{1+\delta_\lambda e^{2s}} = \delta_\lambda e^{2s} + O_L(\delta_\lambda^2).$$

The exact radial residual has a useful normalization

$$\mathcal{N}_g(u) = u^{\frac{4n}{n-4}}\left(\sigma_2(g_u^{-1}A_{g_u}) - \frac{1}{4}\binom{n}{2}\right), \qquad g_u = u^{\frac{4}{n-2}}g.$$

$$\frac{r^n\mathcal{N}_1(u)}{\mathcal{V}^4q^2} = 4(n-1)\frac{\dot p + 2aq}{q} - \frac{1}{4}\binom{n}{2}\frac{\mathcal{V}^{16/(n-4)}}{q^2}, \qquad \mathcal{N}_1 := \mathcal{N}_{g_E}.$$

$$p = \delta_\lambda F + O_L(\delta_\lambda^2), \quad q = \delta_\lambda F + O_L(\delta_\lambda^2), \quad \mathcal{V}^{16/(n-4)} = 16\delta_\lambda^2 e^{4s}(1+O_L(\delta_\lambda)).$$

$$\frac{r^n\mathcal{N}_1(u)}{\mathcal{V}^4q^2} = 4(n-1)\left(\frac{F'}{F} + 2a - \frac{n}{2}\frac{e^{4s}}{F^2}\right) + O_L(\delta_\lambda).$$

Transition: F_γ removes the leading residual

Choose F so that the leading residual is zero

$$\frac{F'}{F} + 2a = \frac{n}{2}\frac{e^{4s}}{F^2} \qquad \Longleftrightarrow \qquad (F^2)' + 4aF^2 = ne^{4s}.$$

$$F_\gamma(s) = \left(\gamma e^{-4as} + e^{4s}\right)^{1/2}, \qquad F_\gamma \sim \sqrt{\gamma}\, e^{-2as}\ (s \ll 0), \qquad F_\gamma \sim e^{2s}\ (s \gg 0).$$

Cut off only near the endpoints; tune γ for first-order matching

$$F^{(0)}_{L,\gamma} = \chi_{\Sigma,L} e^{-2as} + \chi_{B,L} e^{2s} + \omega_L F_\gamma(s).$$

$$\mathfrak{a}_L(\gamma) = -e^{2aL} - ae^{2L} + 2a \int_{-L}^{L} F^{(0)}_{L,\gamma}(s)\, ds, \qquad \mathfrak{a}_L(\gamma_L) = 0, \qquad |\gamma_L - 1| \le Ce^{-2aL}.$$

Add only a tiny exact correction, then integrate the slope

$$p_{\lambda,L} = \chi_{\Sigma,L} p_\Sigma + \chi_{B,L} p_B + \omega_L \delta_\lambda F_{\gamma_L} + \kappa_\lambda \zeta, \qquad |\kappa_\lambda| + |\lambda \partial_\lambda \kappa_\lambda| \le C_L \delta_\lambda^2.$$

$$\widehat{T}_\lambda(R^+_\lambda e^s) = S_\lambda(R^+_\lambda e^{-L}) \exp\left(-2a \int_{-L}^{s} p_{\lambda,L}(\tau)\, d\tau\right), \qquad T_\lambda = \mathcal{K}\widehat{T}_\lambda.$$

$$T_\lambda = U_{\lambda,0}\ (s \simeq -L), \qquad T_\lambda = S_\lambda\ (s \simeq L), \qquad \left\| \frac{r^n \mathcal{N}_1(T_\lambda)}{\mathcal{V}^4 q^2} \right\|_{C^\alpha_{\mathrm{cyl}}} \le Ce^{-cL} + C_L \delta_\lambda = o(1).$$

Finite-dimensional reduction I: solve only the transverse equation

Approximate family and residual

$$p = (\lambda', \xi') \in (0,\infty) \times \mathbb{R}^n, \qquad \bar{v}_p = \text{bubble cores} + \text{Schwarzschild neck} + T_\lambda,$$

$$R_p := \mathcal{N}_g(\bar{v}_p), \qquad v_{p,w} := \bar{v}_p e^{aw}, \qquad a = \frac{n-4}{4},$$

$$\mathcal{N}_g(v_{p,w}) = R_p + \mathscr{L}_p w + Q_p(w).$$

The only directions not solved by the inverse

$$Z_0 = \lambda \partial_\lambda \bar{v}_p, \qquad Z_j = \partial_{\xi_j} \bar{v}_p \quad (1 \le j \le n), \qquad \Psi_a \simeq \mathscr{L}_p Z_a,$$

$$\boxed{\mathscr{L}_p w + R_p + Q_p(w) = \sum_{a=0}^{n} c_a \Psi_a, \qquad \ell_b(w) = 0 \quad (0 \le b \le n).}$$

The PDE is solved in all infinite-dimensional directions; the vector $c = (c_0, \dots, c_n)$ is the remaining finite-dimensional error.

Finite-dimensional reduction II: projected inverse and fixed point

For the talk, keep only the schematic norms

$$X \simeq C^{2,\alpha}, \qquad Y \simeq C^{\alpha}, \qquad \|w\|_X \sim \|w\|_{C^{2,\alpha}}, \quad \|F\|_Y \sim \|F\|_{C^{\alpha}}.$$

Uniform projected inverse

$$\mathscr{L}_p w = F + \sum_{a=0}^{n} c_a \Psi_a, \qquad \ell_b(w) = 0,$$
$$\|w\|_X + |c| \le C\|F\|_Y.$$

Nonlinear equation

$$\|Q_p(w_1) - Q_p(w_2)\|_Y \le C(\|w_1\|_X + \|w_2\|_X)\|w_1 - w_2\|_X,$$
$$w = -G_p\Big(R_p + Q_p(w)\Big), \qquad \|R_p\|_Y = o(1) \Longrightarrow \|w_p\|_X = o(1).$$

Finite-dimensional reduction III: the obstructions are gradients

Projected solution for every parameter

$$v_p=\bar v_p e^{aw_p},\qquad \mathcal{N}_g(v_p)=\sum_{a=0}^{n} c_a(p)\Psi_{a,p},$$

$$\mathscr{J}_N(p):=\mathscr{E}_g(v_p),\qquad p=(\lambda',\xi')\in(0,\infty)\times\mathbb{R}^n.$$

Variationality turns finite-dimensional error into an energy gradient

$$\partial_{p_b}\mathscr{J}_N(p)=4\int_{\mathbb{S}^n}\mathcal{N}_g(v_p)v_p^{-1}\partial_{p_b}v_p\,dv_g=4\sum_{a=0}^{n} c_a(p)M^N_{ab}(p),$$

$$M^N(p)=2aI_{n+1}+o(1).$$

Conclusion of the Lyapunov–Schmidt reduction

$$\nabla\mathscr{J}_N(p_*)=0\Longrightarrow c_0(p_*)=\cdots=c_n(p_*)=0\Longrightarrow \mathcal{N}_g(v_{p_*})=0,\qquad g_{v_{p_*}}\in\Gamma_2^+.$$

The reduced energy: one explicit finite-dimensional model

After solving the transverse equation, only parameters remain

$$\mathscr{I}_N(\xi,\lambda)=\mathscr{E}_2\Big(\bar u_{N,\xi,\lambda}+\phi_{N,\xi,\lambda}\Big),\qquad \|\phi_{N,\xi,\lambda}\|_{C^{2,\alpha}}=o(1),$$

$$s_N=\mu_N^2\epsilon_N^{20}=\epsilon_N^{\frac{n}{2}+8},\qquad \mathscr{I}_N=2\mathscr{E}_2(S^n)+2s_N\mathcal{F}(\xi',\lambda')+o_{C^1}(s_N),$$

$$\nabla_{\xi,\lambda}\mathscr{I}_N=0\quad\Longleftarrow\quad\nabla_{\xi',\lambda'}\mathcal{F}(\xi',\lambda')=o(1).$$

The limit model is controlled by three explicit polynomials

$$\mathcal{F}(0,\sqrt{t})=|W|^2P_f(t),\qquad D^2_{\xi'\xi'}\mathcal{F}(0,\sqrt{t})=|W|^2Q_f(t)I+R_f(t)\mathscr{W},$$

$$\text{target at } (\xi',\lambda')=(0,1):$$

$$P_f'(1)=0,\qquad P_f(1)<0,\qquad P_f''(1)>0,$$

$$Q_f(1)>0,\qquad Q_f(1)+\frac{1}{4}R_f(1)>0.$$

A quartic certificate produces the critical point for $n \geq 27$

Tune one coefficient to solve the scale equation

$$f_a(s) = a - 5960s + 1181s^2 - 73s^3 + s^4,$$

$$\widehat{P}'_{f_a}(1) = \frac{A_n a^2 + B_n a + C_n}{65536(n-3)(n-2)^2}, \qquad a_0(n) = \frac{-B_n - \sqrt{\Delta_n}}{2A_n},$$

$$A_n < 0,$$
$$\Delta_n > 0,$$
$$a_0(n) > 0 \qquad (n \geq 27).$$

Then every required sign becomes an explicit positivity check

$$n = z + 27: \quad \text{all remaining numerators have positive coefficients in } z,$$

$$P'_{f_n}(1) = 0, \qquad P_{f_n}(1) < 0, \qquad P''_{f_n}(1) > 0,$$

$$Q_{f_n}(1) > 0, \qquad Q_{f_n}(1) + \frac{1}{4}R_{f_n}(1) > 0,$$

$$(0, 1) \text{ is a strict nondegenerate minimum of } \mathcal{F} \quad \Longrightarrow \quad \nabla \mathscr{J}_N = 0.$$